**MATH 223.3 2006-2007 (03)**
**TEST #1 - Solutions**
**Integer scores *only*.**

**PART B – 5 points each – total 20**

**B1.** $dx/dt = e^t$, $dy/dt = e^t(\sin t + \cos t)$, $dz/dt = e^t(\cos t - \sin t)$

so $v = \sqrt{(dx/dt)^2 + (dy/dt)^2 + (dz/dt)^2} = e^t\sqrt{3}$ and therefore $s = \sqrt{3}\,(e^t - 1)$

Now, $dx/ds = (dx/dt)(dt/ds) = (1/v)(dx/dt)$ and similarly for $dy/ds$ and $dz/ds$ so that

$$\sqrt{(dx/ds)^2 + (dy/ds)^2 + (dz/ds)^2} = \sqrt{(1/v)^2\,(dx/dt)^2 + (dy/dt)^2 + (dz/dt)^2}$$
$$= (1/v)\sqrt{(dx/dt)^2 + (dy/dt)^2 + (dz/dt)^2} = (1/v)\,v = 1$$

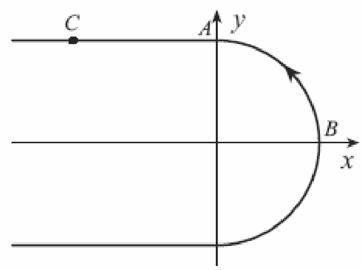
and therefore the vector $(dx/ds, dy/ds, dz/ds)$ is indeed a unit tangent vector along the curve in the direction of increasing $t$, and thus it coincides with $\mathbf{T}$.

**B2.**

Suppose $s$ is the constant speed of the train, and we take $t = 0$ when the train passes through $B$. Along $BA$, $x = R\cos\omega t$, $y = R\sin\omega t$, where $\omega = s/R$. The train passes through $A$ at $t = \pi R/(2s)$. The acceleration of the train along $BA$ is

$$\mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = (-R\omega^2\cos\omega t)\hat{\mathbf{i}} + (-R\omega^2\sin\omega t)\hat{\mathbf{j}}.$$

Consequently, $\displaystyle\lim_{t\to\pi R/(2s)^-} \mathbf{a}(t) = -R\omega^2\hat{\mathbf{j}}$.




Along $AC$, $x = -s[t - \pi R/(2s)]$, $y = R$, so that acceleration on this part of the track is $\mathbf{a} = \mathbf{0}$. Hence $\mathbf{a}$ is discontinuous at $A$.

As an alternative solution, it would be perfectly correct to do the following. Equations are not necessary to gain full marks, *but a proper justification using mathematical expressions is necessary*.

Along the circular and straight segments the speed $v$ is constant and hence the tangential component of the acceleration, $dv/dt$ is the zero vector. Along the straight segment there is no normal component of the acceleration, and obviously the acceleration vector along the straight segment is the zero vector. But, along the circular segment, the centripetal acceleration has magnitude $v^2/R$ and is clearly *not* zero; the acceleration vector along the circular segment is therefore a non-zero vector pointing toward the centre of the circle. At a point where the straight and circular segments join together, the magnitude of the acceleration vector jumps in value and therefore the acceleration vector is *discontinuous*.

The following puts the problem in a perhaps familiar context; it is *not* part of a solution.
Imagine yourself behind the wheel of a car driving around a circular segment, in the clockwise direction, at a constant speed. The driver's side (right) door presses on your left shoulder to keep you following a circular path, but at the instant that you straighten the wheel – instantly! – you will be flung *to your right* – toward the instantaneous centre of curvature (the centre) of the circular segment. That sudden impulse to the right signals the presence of a motion discontinuous in the force (and hence the acceleration) acting upon you.

**B3.**

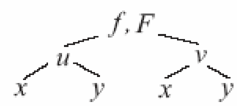
To find the slope $dy/dx$ of the tangent line to $C$ at any point $(x, y)$, we implicitly differentiate $F(x, y) = 0$ with respect to $x$. This can be accomplished by differentiating $F(x, y)$ partially with respect to $x$, and adding to this the partial derivative with respect to $y$ multiplied by $dy/dx$,

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0.$$

Thus, $\dfrac{dy}{dx} = -\dfrac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}$, and it follows that a tangent vector to $C$ at $(x, y)$ is $\left(\dfrac{\partial F}{\partial y}, -\dfrac{\partial F}{\partial x}\right)$. A vector perpendicular to $C$ must therefore be $\left(\dfrac{\partial F}{\partial x}, \dfrac{\partial F}{\partial y}\right)$. Since this is $\nabla F$, the proof is complete.

**B4.**

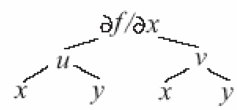
The schematic to the right describes the functional situation $f(x, y) = F[u(x, y), v(x, y)]$ where $u = u(x, y) = (x + y)/2$ and $v = v(x, y) = (x - y)/2$. It gives




$$\frac{\partial f}{\partial x} = \frac{\partial F}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial x} = \frac{1}{2}\frac{\partial F}{\partial u} + \frac{1}{2}\frac{\partial F}{\partial v}.$$

The schematic for $\partial f/\partial x$ leads to

$$\begin{aligned} \frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial u}\left(\frac{\partial f}{\partial x}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial f}{\partial v}\right)\frac{\partial v}{\partial x} \\ &= \left(\frac{1}{2}\frac{\partial^2 F}{\partial u^2} + \frac{1}{2}\frac{\partial^2 F}{\partial u \partial v}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{2}\frac{\partial^2 F}{\partial v \partial u} + \frac{1}{2}\frac{\partial^2 F}{\partial v^2}\right)\left(\frac{1}{2}\right) \\ &= \frac{1}{4}\left(\frac{\partial^2 F}{\partial u^2} + 2\frac{\partial^2 F}{\partial u \partial v} + \frac{\partial^2 F}{\partial v^2}\right). \end{aligned}$$




Similarly, $\dfrac{\partial^2 f}{\partial y^2} = \dfrac{1}{4}\left(\dfrac{\partial^2 F}{\partial u^2} - 2\dfrac{\partial^2 F}{\partial u \partial v} + \dfrac{\partial^2 F}{\partial v^2}\right)$. Hence, $\quad 0 = \dfrac{\partial^2 f}{\partial x^2} - \dfrac{\partial^2 f}{\partial y^2} = \dfrac{\partial^2 F}{\partial u \partial v}$.